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TWO-SIDED ESTIMATES IN UNILATERAL ELASTICITY

PIERO VILLAGGIO
Universitá di Pisa, Pisa, Italy

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Abstract—The paper presents an extension to unilateral problems of the classical method of bounding (above and below) the solutions of linear self-adjoint boundary value problems. Using this extension the solution of the general unilateral problem in linear elasticity is bounded in energy by two suitable defined admissible states belonging to two complementary convex sets.

1. INTRODUCTION

The aim of this paper is to show how some classical methods previously used to bound the energy of solutions to linear self-adjoint boundary value problems may be extended to problems with unilateral constraints.

For such problems, which arise, for instance, in many different parts of mechanics, the question of existence, uniqueness and continuous dependence of the solution on the data have already been rigorously treated by means of the theory of variational inequalities (see, e.g. Lious and Stampacchia [9]). However, in linear problems at least, it is also of some importance to obtain approximations to the solutions. This has usually be done in the bilateral case by bounding some mean quantities of the solution, like the energy, and then using Green's function to obtain pointwise estimates for the solution itself.

The classical techniques essentially derive from a dual interpretation of the minimum principles in the variational calculus as noted by Friedrichs [1]. Later, Diaz and Greenberg [2], Diaz [3], Prager and Synge [4], and Synge [5, 6], systematically extended Friedrichs' analysis to linear elastostatic problems. The solution is equivalent to determining the unique point of intersection of two linear orthogonal subspaces corresponding to the class of kinematically possible configurations and the class of equilibrium configuration, respectively. By choosing two arbitrary elements separately located in each subspace, it is possible to find upper and lower bounds on the strain energy. When a fundamental solution is known, it is also possible to determine pointwise bounds on the elastic displacement and its gradients (see Synge [6]).

As regards unilateral problems, where these can be expressed by a variational inequality, Velte [7] has shown that the classical techniques can be simply extended to obtain bounds for the solutions. Now, however the solution is determined by the point of intersection of two convex subspaces, which are not necessarily uniquely defined. Nevertheless for the sake of uniformity with the bilateral problem, in what follows, we shall continue to associate the elements in these subspaces with the kinematical and equilibrium conditions of the problem. We shall see that such conditions are no longer governed by equations, but rather by equations and inequalities. Moreover, the convex subspaces are not orthogonal. Apart from these features, the general procedure for finding bounds is the same as in the bilateral problems. We construct elements in both convex subspaces to bound the energy and then use two singular states together with Betti's theorem to obtain pointwise bounds on the solution.

The paper consists of three parts. The first describes the existence analysis and also the decomposition of the boundary value problem into the two complementary convex subspaces. In the second part, approximations are found for the strain energy and for the solutions at a point. The last part contains two applications of the general results.

2. NOTATION AND DEFINITIONS

Let $\kappa(B)$ denote the reference configuration of an elastic body which is also taken to be a natural state. The region $\kappa(B)$ is an open, connected, and bounded set of three-dimensional euclidean space, whose boundary $\partial\kappa(B)$ is piecewise regular.† Let $\mathbf{X}$ denote the place of each particle of the body in $\kappa(B)$ and let the state of deformation with respect to $\kappa(B)$ be described by

†That is, decomposable into a finite number of non-overlapping differentiable surfaces.

the displacement vector field $\mathbf{u}(\mathbf{X})$. The tensor field

$$\tilde{\mathbf{E}} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \tag{2.1}$$

where $\nabla \mathbf{u}$ is the displacement gradient, represents the infinitesimal strain.

We shall assume throughout that the displacement and the strain are infinititesimal so that all the conditions of the classical linear theory of elasticity hold. In particular the stress, denoted by the symmetric tensor field $\mathbf{T}$, is related to the linear strain through the expression

$$\mathbf{T} = \mathbf{L}(\tilde{\mathbf{E}}), \tag{2.2}$$

where $\mathbf{L}$ is the elastic tensor. We suppose that $\mathbf{L}$ is symmetric in the sense that

$$L_{ijkl} = L_{jikl} = L_{ijlk} \tag{2.3}$$

for components L_{ijkl} of $\mathbf{L}$ with respect to a given orthonormal base, and moreover that the additional condition

$$L_{ijkl} = L_{klij} \tag{2.4}$$

is satisfied. It is known[8] that (2.4) ensures the existence of a stored energy function.

We suppose that a body force field $\rho \mathbf{b}$ is assigned in $\boldsymbol{\kappa}(B)$, while on part $\overline{\partial_1 \boldsymbol{\kappa}(B)}$ of the boundary the displacement $\hat{\mathbf{u}}$ is prescribed and the traction $\hat{\mathbf{t}}$ is given on the remainder, $\partial_2 \boldsymbol{\kappa}(B)$.

Thus, corresponding to the given data, the elastic state consists of the fields $\mathcal{S} \equiv [\mathbf{u}, \tilde{\mathbf{E}}, \mathbf{T}]$ which are the solution of the boundary value problem:

$$\left.\begin{aligned} \operatorname{div} \mathbf{T} + \rho \mathbf{b} = \mathbf{O} \quad &\text{in} \quad \boldsymbol{\kappa}(B), \\ \mathbf{u} = \hat{\mathbf{u}} \quad &\text{on} \quad \overline{\partial_1 \boldsymbol{\kappa}(B)}, \\ \mathbf{T}\mathbf{n} = \hat{\mathbf{t}} \quad &\text{on} \quad \partial_2 \boldsymbol{\kappa}(B). \end{aligned}\right\} \tag{2.5}$$

where $\mathbf{n}$ is the unit outwards normal on $\partial_2 \boldsymbol{\kappa}(B)$.

When $\overline{\partial_1 \boldsymbol{\kappa}(B)}$ is empty, the external loads must satisfy the conditions of global equilibrium:

$$\int_{\boldsymbol{\kappa}(B)} \rho \mathbf{b} \, dv + \int_{\partial \boldsymbol{\kappa}(B)} \hat{\mathbf{t}} \, ds = \mathbf{O}, \tag{2.6}$$

$$\int_{\boldsymbol{\kappa}(B)} \mathbf{r} \times \rho \mathbf{b} \, dv + \int_{\partial \boldsymbol{\kappa}(B)} \mathbf{r} \times \hat{\mathbf{t}} \, ds = \mathbf{O}, \tag{2.7}$$

where $\mathbf{r} = \mathbf{X} - \mathbf{O}$ and $\mathbf{O}$ is a fixed point. In this case the following normalization conditions must be imposed:

$$\int_{\boldsymbol{\kappa}(B)} \mathbf{u} \, dv = \int_{\boldsymbol{\kappa}(B)} \mathbf{r} \times \mathbf{u} \, dv = \mathbf{O}, \tag{2.8}$$

to exclude arbitrary rigid body motions.

The system (2.5) describes the basic boundary value problem of mixed type of classical elastostatic. Since the components of the elastic tensor satisfy the symmetry condition (2.4), the governing differential operator is formally selfadjoint.

In the numerical treatement of problems, or in discussing *a priori* estimates on the solutions, it is often convenient to formulate problem (2.5) in terms of variational principles. Many of these results are needed in the extension to unilateral problems of similar techniques, therefore we shall briefly sketch the main steps of the development leading to them.

We consider first the set of the **kinematically admissible** elastic states, satisfying the displacement-deformations, the stress-deformation relations, and the displacements boundary conditions.

We denote these states by $\mathcal{S}' \equiv [\mathbf{u}', \tilde{\mathbf{E}}', \mathbf{T}']$ and observe that they can be regarded as elements of a Hilbert space endowed with the (symmetric) scalar product

$$((\mathcal{S}', \mathcal{S}')) = \int_{\kappa(B)} tr(\mathbf{T}'\tilde{\mathbf{E}}')\,dv = \int_{\kappa(B)} tr\{\mathsf{L}[\tilde{\mathbf{E}}']\tilde{\mathbf{E}}'\}\,dv. \tag{2.9}$$

The admissible states $\mathcal{S}'$ describe a certain linear manifold $\mathcal{L}'$, which we will call *the manifold of kinematically admissible configurations.* We can decompose $\mathcal{S}'$ into the sum

$$\mathcal{S}' = \mathcal{S}'_0 + \mathcal{T}', \tag{2.10}$$

where $\mathcal{S}'_0$ is fixed element satisfying the non-homogeneous boundary conditions on $\overline{\partial_1\kappa(B)}$ and $\mathcal{T}'$ is a variable element vanishing on $\overline{\partial_1\kappa(B)}$.

Similarly, we will denote by $\mathcal{S}'' \equiv [\mathbf{u}'', \tilde{\mathbf{E}}'', \mathbf{T}'']$ the **statically admissible** states which satisfy the equilibrium equations, the stress-deformation relations, and the traction boundary conditions.

The states $\mathcal{S}''$ also describe a linear manifold $\mathcal{L}''$ in a Hilbert space, for which the (symmetric) scalar product becomes

$$((\mathcal{S}'', \mathcal{S}'')) = \int_{\kappa(B)} tr(\mathbf{T}''\tilde{\mathbf{E}}'')\,dv = \int_{\kappa(B)} tr\{\mathbf{T}''\mathsf{L}^{-1}[\mathbf{T}'']\}\,dv, \tag{2.11}$$

where L^{-1} is the inverse of the elastic tensor.† We call $\mathcal{L}''$ *the manifold of statically admissible configurations.*

The state $\mathcal{S}''$ may also be decomposed into the sum

$$\mathcal{S}'' = \mathcal{S}''_0 + \mathcal{T}'', \tag{2.12}$$

where $\mathcal{S}''_0$ is the fixed element satisfying the non-homogeneous equilibrium equations in $\kappa(B)$ and the non-homogeneous boundary conditions on $\partial_2\kappa(B)$, and $\mathcal{T}''$ is a variable element satisfying the corresponding homogeneous equations and boundary conditions.

It is easy to verify that $\mathcal{L}'$ and $\mathcal{L}''$ are orthogonal with respect to the scalar products (2.9) and (2.11). In fact, applying the Gauss–Green formula to the two states $\mathcal{T}'$ and $\mathcal{T}''$, belonging respectively to $\mathcal{L}'$ and $\mathcal{L}''$, we obtain

$$((\mathcal{T}', \mathcal{T}'')) = \int_{\kappa(B)} tr(\mathbf{T}''\tilde{\mathbf{E}}')\,dv = -\int_{\kappa(B)} \operatorname{div}\mathbf{T}''\cdot\mathbf{u}'\,dv + \int_{\partial\kappa(B)} \mathbf{T}''\mathbf{n}\cdot\mathbf{u}'\,ds.$$

The right side vanishes, because, by hypothesis,

$$\operatorname{div}\mathbf{T}'' = \mathbf{O} \quad \text{in} \quad \kappa(B),$$
$$\mathbf{T}''\mathbf{n} = \mathbf{O} \quad \text{on} \quad \partial_2\kappa(B), \quad \text{and} \quad \mathbf{u}' = \mathbf{O} \quad \text{on} \quad \overline{\partial_1\kappa(B)}.$$

If $\mathcal{S}'$ and $\mathcal{S}''$ are allowed to vary in the manifolds $\mathcal{L}'$ and $\mathcal{L}''$, the solution of problem (2.5), which is obviously the unique element of intersection of the two manifolds, may be regarded as the field $\mathcal{S}$ which is solution to the following minimum problem:

$$\|\mathcal{S}' - \mathcal{S}''\|^2 = ((\mathcal{S}' - \mathcal{S}'', \mathcal{S}' - \mathcal{S}'')) = \min \tag{2.13}$$

for any $\mathcal{S}' \in \mathcal{L}'$, $\mathcal{S}'' \in \mathcal{L}''$.

Direct consequences of (2.13) intuitively obvious from Fig. 1, are the relations [6]

$$((\mathcal{S} - \mathcal{S}', \mathcal{S} - \mathcal{S}'')) = 0, \tag{2.14}$$

†$\mathsf{L}^{-1} = \mathsf{K}$ is also called **compliance tensor** (see Gurtin [8]).

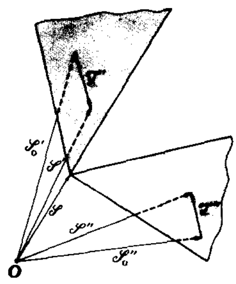

Fig. 1.

$$\|\mathscr{S}' - \mathscr{S}''\|^2 = \|\mathscr{S}' - \mathscr{S}\|^2 + \|\mathscr{S}'' - \mathscr{S}\|^2. \tag{2.15}$$

Formulae (2.14) and (2.15) are crucial in the provision of *a priori* bounds for the solution of problem (2.5), or, more precisely, for the bound on the norm $\|\mathscr{S}\|^2$ which is twice the energy of deformation. In fact, they are equivalent to the single equation

$$\left\|\mathscr{S} - \frac{1}{2}(\mathscr{S}' + \mathscr{S}'')\right\|^2 = \left\|\frac{1}{2}(\mathscr{S}' - \mathscr{S}'')\right\|^2, \tag{2.16}$$

which proves that the vector $\mathscr{S}$ lies on the sphere with centre $\mathscr{C}_0 = (1/2)(\mathscr{S}' + \mathscr{S}'')$ and radius $R_0 = (1/2)\|\mathscr{S}' - \mathscr{S}''\|$. In other words, $\mathscr{S}$ satisfies the bounds

$$(\|\mathscr{C}_0\| - R_0)^2 \leq \|\mathscr{S}\|^2 \leq (\|\mathscr{C}_0\| + R_0)^2. \tag{2.17}$$

These bounding formulae undergo a radical simplification when the boundary datum is homogeneous; in this case one of the two manifolds $\mathscr{L}'$ or $\mathscr{L}''$ contains the origin. To fix our ideas, let us consider, the case when $\mathscr{L}''$ contains the origin (Fig. 2). Then, if $\mathscr{S}'$ and $\mathscr{S}''$ are two admissible states, the state $\kappa\mathscr{S}''$, with κ constant, is also admissible.

Putting $\kappa = 0$, and therefore $\mathscr{S}'' = \mathbf{O}$, in (2.17), we obtain directly the upper bound

$$\|\mathscr{S}\|^2 \leq \|\mathscr{S}'\|^2. \tag{2.18}$$

Conversely, when we put $\kappa\mathscr{S}''$ for $\mathscr{S}''$ into the lower bound in (2.17), it becomes a function of κ which reaches its maximum for

$$\kappa = \frac{((\mathscr{S}', \mathscr{S}''))}{\|\mathscr{S}''\|^2}.$$

Thus, on substituting this value into the left hand side of (2.17), we obtain the lower bound

$$\|\mathscr{S}\| \geq \frac{((\mathscr{S}', \mathscr{S}''))^2}{\|\mathscr{S}''\|^2} \tag{2.19}$$

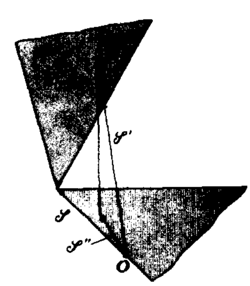

Fig. 2.

The same argument applies (apart from the obvious transposition of $\mathscr{L}'$ with $\mathscr{L}''$) to the case when $\mathscr{L}'$ contains the origin.

From the bound for the energy we can calculate pointwise bounds for the elastic displacement using the Green formula. We denote by $\mathscr{S}_{\mathbf{Y}}[\mathbf{l}](\mathbf{X})$ the Kelvin state corresponding to a load $\mathbf{l}$ concentrated at the point $\mathbf{Y}$.

We write [8]

$$\mathscr{S}_{\mathbf{Y}}[\mathbf{l}] \equiv [\mathbf{u}_{\mathbf{Y}}[\mathbf{l}], \tilde{\mathbf{E}}_{\mathbf{Y}}[\mathbf{l}], \mathbf{T}_{\mathbf{Y}}[\mathbf{l}]]$$

to indicate the elastic state, defined as the limit of a sequence $\{\rho \mathbf{b}_m\}$ of fields of body forces tending to a concentrated load $\mathbf{l}$ in $\mathbf{Y}$ with the following properties:

(1) In $\boldsymbol{\kappa}(B)-\{\mathbf{Y}\}$, $\mathscr{S}_{\mathbf{Y}}[\mathbf{l}]$ corresponds to null body forces;
(2) $\mathbf{u}_{\mathbf{Y}}[\mathbf{l}] = 0(r^{-1})$ and $\mathbf{T}_{\mathbf{Y}}[\mathbf{l}] = 0(r^{-2})$ when r ($r = |\mathbf{X}-\mathbf{Y}|$) tends to zero or to infinity;
(3) for every sphere Σ_η, of radius η and center at $\mathbf{Y}$, the equations

$$\int_{\partial\Sigma_\eta} \mathbf{T}_{\mathbf{Y}}[\mathbf{l}]\mathbf{n}\, ds = \mathbf{l}, \quad \int_{\partial\Sigma_\eta} \mathbf{r}\times \mathbf{T}_{y}[\mathbf{l}]\mathbf{n}\, ds = \mathbf{O}$$

hold, where $\partial\Sigma_\eta$ is the surface of Σ_η;
(4) for each vector $\mathbf{v}$

$$\mathbf{l}\cdot \mathbf{u}_{\mathbf{Y}}[\mathbf{v}] = \mathbf{v}\cdot \mathbf{u}_{\mathbf{Y}}[\mathbf{l}],$$
$$\mathbf{T}_{\mathbf{Y}}[\mathbf{v}]\mathbf{l} = \mathbf{T}^{\ddagger}_{\mathbf{Y}}[\mathbf{l}]\mathbf{v},$$

where $\mathbf{T}^{\ddagger}_{\mathbf{Y}}$ is a uniquely defined stress field, called the *adjoint* field of $\mathbf{T}_{\mathbf{Y}}$ [8].

Now let us apply the Betti theorem to the region $\boldsymbol{\kappa}(B)-\Sigma_\eta$, conbining the solution of the problem (2.5) with the Kelvin state. We write

$$\int_{\boldsymbol{\kappa}(B)-\Sigma_\eta} tr\{\mathbf{T}_{\mathbf{Y}}[\mathbf{l}]\tilde{\mathbf{E}}\}\, dv = \int_{\boldsymbol{\kappa}(B)-\Sigma_\eta} tr\{\mathbf{T}\tilde{\mathbf{E}}_{\mathbf{Y}}[\mathbf{l}]\}\, dv,$$

and transform this expression by an integration by parts. We get easily

$$-\int_{\boldsymbol{\kappa}(B)-\Sigma_\eta} \operatorname{div} \mathbf{T}_{\mathbf{Y}}[\mathbf{l}]\cdot \mathbf{u}\, dv + \int_{\partial\boldsymbol{\kappa}(B)} \mathbf{T}_{\mathbf{Y}}[\mathbf{l}]\mathbf{n}\cdot \mathbf{u}\, ds - \int_{\partial\Sigma_\eta} \mathbf{T}_{\mathbf{Y}}[\mathbf{l}]\mathbf{n}\cdot \mathbf{u}\, ds$$
$$= -\int_{\boldsymbol{\kappa}(B)-\Sigma_\eta} \operatorname{div} \mathbf{T}\cdot \mathbf{u}_{\mathbf{Y}}[\mathbf{l}]\, dv + \int_{\partial\boldsymbol{\kappa}(B)} \mathbf{T}\mathbf{n}\cdot \mathbf{u}_{\mathbf{Y}}[\mathbf{l}]\, ds - \int_{\partial\Sigma_\eta} \mathbf{T}\mathbf{n}\cdot \mathbf{u}_{\mathbf{Y}}[\mathbf{l}]\, ds,$$

whence, using the properties of the Kelvin state, and taking the limit as $\eta \to 0$, we arrive at the formula (of Somigliana)

$$\mathbf{u}(\mathbf{Y})\cdot \mathbf{l} = \int_{\boldsymbol{\kappa}(B)} \rho \mathbf{b}\cdot \mathbf{u}_{\mathbf{Y}}[\mathbf{l}]\, dv + \int_{\partial\boldsymbol{\kappa}(B)} (\mathbf{T}\mathbf{n}\cdot \mathbf{u}_{\mathbf{Y}}[\mathbf{l}] - \mathbf{T}_{\mathbf{Y}}[\mathbf{l}]\mathbf{n}\cdot \mathbf{u})\, ds. \tag{2.20}$$

Finally, applying the property (4) of the Kelvin state, we find that (2.20) is equivalent to the following formula

$$\mathbf{u}(\mathbf{Y}) = \int_{\boldsymbol{\kappa}(B)} \mathbf{u}_{\mathbf{Y}}[\rho \mathbf{b}]\, dv + \int_{\partial\boldsymbol{\kappa}(B)} (\mathbf{u}_{\mathbf{Y}}[\mathbf{T}\mathbf{n}] - \mathbf{T}^{\ddagger}_{\mathbf{Y}}[\mathbf{u}]\mathbf{n})\, ds. \tag{2.21}$$

In the right hand side of (2.21) some quantities are known and others unknown. Thus, we see that

$$\mathbf{U}_{\mathbf{Y}} = \int_{\boldsymbol{\kappa}(B)} \mathbf{u}_{\mathbf{Y}}[\rho \mathbf{b}]\, dv + \int_{\partial_2\boldsymbol{\kappa}(B)} \mathbf{u}_{\mathbf{Y}}[\mathbf{T}\mathbf{n}]\, ds - \int_{\partial_1\boldsymbol{\kappa}(B)} \mathbf{T}^{\ddagger}_{\mathbf{Y}}[\mathbf{u}]\mathbf{n}\, ds \tag{2.22}$$

is known, and is expressible in terms of the data of the problem. On the other hand, to evaluate the unknown part in the right hand side of (2.21), we introduce two auxiliary elastic states

$$\mathcal{S}'_{\mathrm{Y}} \equiv [\mathbf{u}'_{\mathrm{Y}}, \tilde{\mathbf{E}}'_{\mathrm{Y}}, \mathbf{T}'_{\mathrm{Y}}], \mathcal{S}''_{\mathrm{Y}} \equiv [\mathbf{u}''_{\mathrm{Y}}, \tilde{\mathbf{E}}''_{\mathrm{Y}}, \mathbf{T}''_{\mathrm{Y}}],$$

defined in the following way:

(1) $\mathbf{u}'_{\mathrm{Y}}$ is a continuous field of displacement such that

$$\mathbf{u}'_{\mathrm{Y}}(\mathbf{X}) = \mathbf{u}_{\mathrm{Y}}(\mathbf{X}) \quad \text{on} \quad \overline{\partial_1 \boldsymbol{\kappa}(B)}; \tag{2.23}$$

(2) $\mathbf{T}''_{\mathrm{Y}}$ is a piecewise continuous stress field such that

$$\operatorname{div} \mathbf{T}''_{\mathrm{Y}} = \mathbf{O} \quad \text{in} \quad \boldsymbol{\kappa}(B), \mathbf{T}''^{*}_{\mathrm{Y}}\mathbf{n} = \mathbf{T}^{*}_{\mathrm{Y}}\mathbf{n} \quad \text{on} \quad \partial_2 \boldsymbol{\kappa}(B); \tag{2.24}$$

(3)

$$\int_{\partial_2 \boldsymbol{\kappa}(B)} \mathbf{u}'_{\mathrm{Y}}[\mathbf{T}\mathbf{n}]\, \mathrm{d}s = \mathbf{O}, \int_{\partial_1 \boldsymbol{\kappa}(B)} \mathbf{T}''_{\mathrm{Y}}[\mathbf{u}]\mathbf{n}\, \mathrm{d}s = \mathbf{O}. \tag{2.25}$$

Then, using these properties of the auxiliary fields, we are able to express (2.21) as

$$\mathbf{u}(\mathbf{Y}) = \mathbf{U}_{\mathrm{Y}} + \int_{\partial \boldsymbol{\kappa}(B)} \mathbf{u}'_{\mathrm{Y}}[\mathbf{T}\mathbf{n}]\, \mathrm{d}s - \int_{\partial \boldsymbol{\kappa}(B)} \mathbf{T}''^{*}_{\mathrm{Y}}[\mathbf{u}]\mathbf{n}\, \mathrm{d}s. \tag{2.26}$$

But, from the divergence theorem, we have

$$\int_{\partial \boldsymbol{\kappa}(B)} \mathbf{u}'_{\mathrm{Y}}[\mathbf{T}\mathbf{n}]\, \mathrm{d}s = \int_{\boldsymbol{\kappa}(B)} tr(\mathbf{T}\tilde{\mathbf{E}}'_{\mathrm{Y}})\, \mathrm{d}v - \int_{\boldsymbol{\kappa}(B)} \mathbf{u}'_{\mathrm{Y}}[\rho \mathbf{b}]\, \mathrm{d}v,$$

$$\int_{\partial \boldsymbol{\kappa}(B)} \mathbf{T}''^{*}_{\mathrm{Y}}[\mathbf{u}]\mathbf{n}\, \mathrm{d}s = \int_{\boldsymbol{\kappa}(B)} tr(\mathbf{T}''_{\mathrm{Y}}\tilde{\mathbf{E}})\, \mathrm{d}v,$$

so that we may write (2.26) in the more classical way

$$\mathbf{u}(\mathbf{Y}) - \mathbf{U}_{\mathrm{Y}} + \int_{\boldsymbol{\kappa}(B)} \mathbf{u}'_{\mathrm{Y}}[\rho \mathbf{b}]\, \mathrm{d}v = ((\mathcal{S}, \mathcal{S}'_{\mathrm{Y}} - \mathcal{S}''_{\mathrm{Y}})). \tag{2.27}$$

We know how to bound the second member of (2.27), because, if $\mathcal{G}$ is an arbitrary elastic state, by the Cauchy–Schwarz inequality, we have

$$\left|\left(\left(\mathcal{S} - \frac{1}{2}(\mathcal{S}' + \mathcal{S}''), \mathcal{G}\right)\right)\right| \le \left\|\mathcal{S} - \frac{1}{2}(\mathcal{S}' + \mathcal{S}'')\right\| \|\mathcal{G}\|,$$

from which, on remembering (2.16), we obtain

$$\left|\left(\left(\mathcal{S} - \frac{1}{2}(\mathcal{S}' + \mathcal{S}''), \mathcal{G}\right)\right)\right| \le \frac{1}{2}\|\mathcal{S}' - \mathcal{S}''\|\, \|\mathcal{G}\|. \tag{2.28}$$

This means that

$$\frac{1}{2}((\mathcal{S}' + \mathcal{S}'', \mathcal{G})) - \frac{1}{2}\|\mathcal{S}' - \mathcal{S}''\|\, \|\mathcal{G}\| \le ((\mathcal{S}, \mathcal{G})) \le \frac{1}{2}((\mathcal{S}' + \mathcal{S}'', \mathcal{G})) + \frac{1}{2}\|\mathcal{S}' - \mathcal{S}''\| \|\mathcal{G}\|,$$

that is, introducing $\mathcal{C}_0$ and R_0, we can write

$$((\mathcal{C}_0, \mathcal{G})) - R_0\|\mathcal{G}\| \le ((\mathcal{S}, \mathcal{G})) \le ((\mathcal{C}_0, \mathcal{G})) + R_0\|\mathcal{G}\|. \tag{2.29}$$

Then (2.29) furnishes the sought bound for the second member of (2.27), after we put

$$\mathcal{G} = \mathcal{S}'_{\mathbf{Y}} - \mathcal{S}''_{\mathbf{Y}}.$$

In a similar manner, on differentiating with respect to $\mathbf{Y}$ both members of (2.21) and choosing suitable auxiliary states, we obtain bounds for the components of stress and deformation.

3. PROPERTIES OF SOLUTIONS OF THE NON-LINEAR PROBLEM

All results described in the previous section are well known[6]. We now propose extending them to solutions of elastostatic problems subject to unilateral constraints.

More precisely, we shall consider problem (2.5) but for solutions subject to a constraint of the type $\mathbf{u} \in \mathfrak{R}_1$, where $\mathfrak{R}_1$ is a convex subset of the Hilbert space, where the problem is formulated. Then, if $\mathcal{M}$ is the linear manifold of functions such that $\mathbf{u} - \hat{\mathbf{u}}$ have null trace on $\overline{\partial_1 \boldsymbol{\kappa}(B)}$, we call $\mathfrak{R}$ the set of admissible functions defined by the relation

$$\mathfrak{R} = \mathcal{M} \cap \mathfrak{R}_1,$$

which is clearly convex with $\mathfrak{R}_1$. Thus the problem with constraints is reduced to seeking a function $\mathbf{u}_{\mathfrak{R}} \in \mathfrak{R}$, which is solution of the variational inequality

$$\int_{\boldsymbol{\kappa}(B)} tr\{\mathbf{T}_{\mathfrak{R}}(\tilde{\mathbf{E}}' - \tilde{\mathbf{E}}_{\mathfrak{R}})\}\, \mathrm{d}v \geq \int_{\boldsymbol{\kappa}(B)} \rho \mathbf{b} \cdot (\mathbf{u}' - \mathbf{u}_{\mathfrak{R}})\, \mathrm{d}v + \int_{\partial_2 \boldsymbol{\kappa}(B)} \hat{\mathbf{t}} \cdot (\mathbf{u}' - \mathbf{u}_{\mathfrak{R}})\, \mathrm{d}s \quad \forall \mathbf{u}' \in \mathfrak{R}. \tag{3.1}$$

It is known that a solution exists to problem (3.1) under the same hypotheses ensuring the existence of solutions for the problem (2.1)[9, 10]. We wish to consider now the question of bounding *a priori* the solutions.

We begin observing that, if $\mathfrak{R} \equiv \mathcal{M}$, the solution of the problem (3.1) coincides with the one for the problem (2.5). If $\mathcal{S} \equiv [\mathbf{u}, \tilde{\mathbf{E}}, \mathbf{T}]$ is this solution, it satisfies the equation

$$((\mathcal{S}, \mathcal{S}')) = \int_{\boldsymbol{\kappa}(B)} tr(\mathbf{T}\mathbf{E}')\, \mathrm{d}v = \int_{\boldsymbol{\kappa}(B)} \rho \mathbf{b} \cdot \mathbf{u}'\, \mathrm{d}v + \int_{\partial_2 \boldsymbol{\kappa}(B)} \hat{\mathbf{t}} \cdot \mathbf{u}'\, \mathrm{d}s \quad \forall \mathbf{u}' \in \mathcal{M}. \tag{3.2}$$

This is the weak formulation of the problem (2.5). Then, denoting with $\mathcal{S}_{\mathfrak{R}}$ the elastic state corresponding to the solution of the inequality (3.1), we can give this inequality the form

$$((\mathcal{S}_{\mathfrak{R}}, \mathcal{S}' - \mathcal{S}_{\mathfrak{R}})) \geq ((\mathcal{S}, \mathcal{S}' - \mathcal{S}_{\mathfrak{R}})) \quad \forall \mathcal{S}' \in \mathfrak{R}. \tag{3.3}$$

Once $\mathcal{S}$ is known, the theory tells us that there exists a unique element $\mathcal{S}_{\mathfrak{R}} \in \mathfrak{R}$ satisfying (3.3).

We will call (3.3) *the variational inequality in the direct form*. However, as Velte[7] has shown, it is possible to formulate the problem in a reciprocal form, which is the natural extension of the method of Friedrichs[1] for bilateral problems. In fact the following theorem (of Velte) holds:

Theorem 3.1 (Velte[7]). *The solution $\mathcal{S}_{\mathfrak{R}}$ of the inequality* (3.3) *is also a solution of the inequality*

$$((\mathcal{S}_{\mathfrak{R}}, \mathcal{S}'' - \mathcal{S}_{\mathfrak{R}})) \geq ((\mathcal{S}, \mathcal{S}'' - \mathcal{S}_{\mathfrak{R}})) \quad \forall \mathcal{S}'' \in \mathfrak{R}_c, \tag{3.4}$$

where $\mathfrak{R}_c$ is the set of all elements $\mathcal{S}''$ satisfying the complementary inequality

$$((\mathcal{S}'', \mathcal{S}' - \mathcal{S}'')) \geq ((\mathcal{S}, \mathcal{S}' - \mathcal{S}'')) \quad \forall \mathcal{S}' \in \mathfrak{R}. \tag{3.5}$$

Proof. The variational inequality (3.3) is equivalent to the minimum problem

$$J(\mathcal{S}') = \frac{1}{2}((\mathcal{S}', \mathcal{S}')) - ((\mathcal{S}, \mathcal{S}')) = \min \qquad \text{for} \quad \mathcal{S}' \in \mathfrak{R}.$$

But, since the bilinear form (2.9) is non-negative, we have

$$\frac{1}{2}((\mathscr{S}',\mathscr{S}')) \geq -\frac{1}{2}((\mathscr{S}'',\mathscr{S}'')) + ((\mathscr{S}',\mathscr{S}'')) \quad \forall \mathscr{S}',\mathscr{S}'' \in H^1(\boldsymbol{\kappa}(B)).$$

In particular, for any $\mathscr{S}' \in \mathfrak{R}$ and $\mathscr{S}'' \in H^1(\boldsymbol{\kappa}(B))$, we may write

$$\begin{aligned} J(\mathscr{S}') &\geq -\frac{1}{2}((\mathscr{S}'',\mathscr{S}'')) + ((\mathscr{S}',\mathscr{S}'')) - ((\mathscr{S},\mathscr{S}')) \\ &= J(\mathscr{S}'') + \{((\mathscr{S}'',\mathscr{S}'-\mathscr{S}'')) - ((\mathscr{S},\mathscr{S}'-\mathscr{S}''))\}. \end{aligned} \tag{3.6}$$

We prove now that this inequality implies that, if $\mathscr{S}_{\mathfrak{R}}$ is the solution of (3.3), then $\mathscr{S}_{\mathfrak{R}} = \mathfrak{R} \cap \mathfrak{R}_c$ and

$$\min_{\mathscr{S}' \in \mathfrak{R}} J(\mathscr{S}') = J(\mathscr{S}_{\mathfrak{R}}) = \max_{\mathscr{S}'' \in \mathfrak{R}_c} J(\mathscr{S}''). \tag{3.7}$$

In fact, an element of $H^1(\boldsymbol{\kappa}(B))$ is solution of (3.3) if and only if it belongs to $\mathfrak{R} \cap \mathfrak{R}_c$, but, as (3.3) has only one solution, it follows that $\mathscr{S}_{\mathfrak{R}} = \mathfrak{R} \cap \mathfrak{R}_c$. In addition, for any pair $\mathscr{S}' \in \mathfrak{R}$ and $\mathscr{S}'' \in \mathfrak{R}_c$, one has from (3.6) $J(\mathscr{S}') \geq J(\mathscr{S}'')$. Since $\mathscr{S}_{\mathfrak{R}} \in \mathfrak{R} \cap \mathfrak{R}_c$ it follows that $J(\mathscr{S}') \geq J(\mathscr{S}_{\mathfrak{R}})$ for any $\mathscr{S}' \in \mathfrak{R}$, and $J(\mathscr{S}_{\mathfrak{R}}) \geq J(\mathscr{S}'')$ for any $\mathscr{S}'' \in \mathfrak{R}_c$. Hence we have (3.7) or alternatively (3.3) and (3.4). □

We observe explicitly that despite its formal resemblance, (3.5) has not a unique solution, while (3.3) has one and only one solution. This is because the solutions $\mathscr{S}''$ of (3.5) do not necessarily lie in $\mathfrak{R}$.

From a knowledge of two arbitrary elements $\mathscr{S}' \in \mathfrak{R}$ and $\mathscr{S}'' \in \mathfrak{R}_c$ we may easily derive a two-sided estimate for the energy of $\mathscr{S}_{\mathfrak{R}}$. Starting indeed from the identity

$$\left\| \mathscr{S}_{\mathfrak{R}} - \frac{1}{2}(\mathscr{S}' + \mathscr{S}'') \right\|^2 = \frac{1}{4}\|\mathscr{S}' - \mathscr{S}_{\mathfrak{R}} + \mathscr{S}'' - \mathscr{S}_{\mathfrak{R}}\|^2, \tag{3.8}$$

we may verify at once that the following inequalities hold for the right hand side of (3.5):

$$\begin{aligned} \frac{1}{4}\|\mathscr{S}' - \mathscr{S}_{\mathfrak{R}} + \mathscr{S}'' - \mathscr{S}_{\mathfrak{R}}\|^2 &\leq \frac{1}{4}(\|\mathscr{S}' - \mathscr{S}_{\mathfrak{R}}\|^2 + \|\mathscr{S}'' - \mathscr{S}_{\mathfrak{R}}\|^2) \\ &= \frac{1}{4}\{\|\mathscr{S}' - \mathscr{S}''\|^2 - 2((\mathscr{S}'' - \mathscr{S}_{\mathfrak{R}}, \mathscr{S}_{\mathfrak{R}} - \mathscr{S}'))\}. \end{aligned} \tag{3.9}$$

From this, using (3.5), we derive

$$\left\| \mathscr{S}_{\mathfrak{R}} - \frac{1}{2}(\mathscr{S}' + \mathscr{S}'') \right\|^2 \leq \frac{1}{4}\|\mathscr{S}' - \mathscr{S}''\|^2, \tag{3.10}$$

which is exactly (2.16), when $\mathscr{S}'$ ranges over $\mathfrak{R}$ and $\mathscr{S}''$ over $\mathfrak{R}_c$. The geometric illustration of this situation is shown in Fig. 3.

Once (3.8) has been provided, (2.17) and (2.28) follow as simple corollaries. On the contrary, it is

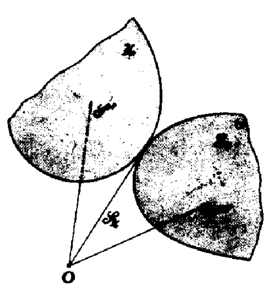

Fig. 3.

not longer possible to derive inequalities of the type (2.19), because $\mathfrak{R}$ and $\mathfrak{R}_c$ are not linear manifolds, so that for admissable $\mathcal{S}''$ (or $\mathcal{S}'$), $\kappa\mathcal{S}''$ (or $\kappa\mathcal{S}'$) is not necessarily admissible.

Remark 3.1 If $\mathcal{S}$, the solution of the unconstrained problem, does not belong to $\mathfrak{R}$, then it belongs to $\mathfrak{R}_c$. This may be seen by taking $\mathcal{S}'' = \hat{\mathcal{S}}$ in (3.5).

Once $\mathfrak{R}$ and $\mathcal{S}$ are known, this allows us to characterize geometrically $\mathfrak{R}_c$, as the locus of vectors $\mathcal{S}''$ such that $\mathcal{S}'' - \mathcal{S}$ makes an acute angle with $\mathcal{S}' - \mathcal{S}''$.

Remark 3.2. The solution of the variational inequality (3.1) is equivalent, geometrically speaking, to identify the element of minimal distance between the convex sets $\mathfrak{R}$ and $\mathfrak{R}_c$.

4. LOCAL BOUNDS ON SOLUTIONS

We have seen how a knowledge of two complementary admissible states, enables the deformation energy to be bounded. We now propose to find pointwise bounds for solutions as is done in bilateral problems. In the present situation, we cannot directly employ the Kelvin state, because the state $\mathcal{S}$ is not a solution to equation (2.5) but to inequality (3.1). Therefore, the value of div $\mathbf{T}_{\mathfrak{R}}$ in $\boldsymbol{\kappa}(B)$ and of $\mathbf{u}_{\mathfrak{R}}$ or $\mathbf{t}_{\mathfrak{R}}$ on $\partial\boldsymbol{\kappa}(B)$ are unknown. As we have seen, these quantities are essential for the determination of the function $\mathbf{U}_\mathbf{Y}$ defined by (2.22).

In any case, we can still apply Betti's theorem to the region $\boldsymbol{\kappa}(B) - \Sigma_\eta$, combining the Kelvin state with the solution of (3.1). We write

$$\int_{\boldsymbol{\kappa}(B)-\Sigma_\eta} tr\{\mathbf{T}_\mathbf{Y}[\mathbf{l}]\tilde{\mathbf{E}}_{\mathfrak{R}}\}\, dv = \int_{\boldsymbol{\kappa}(B)-\Sigma_\eta} tr\{\mathbf{T}_{\mathfrak{R}}\tilde{\mathbf{E}}_\mathbf{Y}[\mathbf{l}]\}\, dv, \tag{4.1}$$

and integrate the first member by parts. We obtain

$$\int_{\partial\boldsymbol{\kappa}(B)} \mathbf{T}_\mathbf{Y}[\mathbf{l}]\mathbf{n}\cdot\mathbf{u}_{\mathfrak{R}}\, ds - \int_{\partial\Sigma_\eta} \mathbf{T}_\mathbf{Y}[\mathbf{l}]\mathbf{n}\cdot\mathbf{u}_{\mathfrak{R}}\, ds - \int_{\boldsymbol{\kappa}(B)-\Sigma_\eta} \operatorname{div}\mathbf{T}_\mathbf{Y}[\mathbf{l}]\cdot\mathbf{u}_{\mathfrak{R}}\, dv$$
$$= \int_{\boldsymbol{\kappa}(B)-\Sigma_\eta} tr\{\mathbf{T}_{\mathfrak{R}}\tilde{\mathbf{E}}_\mathbf{Y}[\mathbf{l}]\}\, dv. \tag{4.2}$$

From here, taking the limit as $\eta \to 0$, we arrive at the equation

$$\mathbf{u}(\mathbf{Y})\cdot\mathbf{l} = -\int_{\partial\boldsymbol{\kappa}(B)} \mathbf{T}_\mathbf{Y}[\mathbf{l}]\mathbf{n}\cdot\mathbf{u}_{\mathfrak{R}}\, ds + \int_{\boldsymbol{\kappa}(B)} tr\{\mathbf{T}_{\mathfrak{R}}\tilde{\mathbf{E}}_\mathbf{Y}[\mathbf{l}]\}\, dv,$$

which is equivalent to

$$\mathbf{u}(\mathbf{Y}) = -\int_{\partial\boldsymbol{\kappa}(B)} \mathbf{T}_\mathbf{Y}^\ddagger[\mathbf{u}_{\mathfrak{R}}]\mathbf{n}\, ds + \int_{\boldsymbol{\kappa}(B)} tr(\mathbf{T}_{\mathfrak{R}}\tilde{\mathbf{E}}_\mathbf{Y})\, dv. \tag{4.3}$$

We introduce now an auxiliary field

$$\mathcal{S}_\mathbf{Y}'' \equiv [\mathbf{u}_\mathbf{Y}'', \tilde{\mathbf{E}}_\mathbf{Y}'', \mathbf{T}_\mathbf{Y}''],$$

with the properties that

(1) $\operatorname{div}\mathbf{T}_\mathbf{Y}'' = \mathbf{O}$ in $\boldsymbol{\kappa}(B)$,

(2) $\mathbf{T}_\mathbf{Y}''[\mathbf{l}]\mathbf{n} = \mathbf{T}_\mathbf{Y}[\mathbf{l}]$ on $\partial\boldsymbol{\kappa}(B)$.

Thus (4.3) becomes

$$\mathbf{u}(\mathbf{Y}) = \int_{\boldsymbol{\kappa}(b)} tr(\mathbf{T}_{\mathfrak{R}}\tilde{\mathbf{E}}_\mathbf{Y} - \mathbf{T}_\mathbf{Y}''\tilde{\mathbf{E}}_{\mathfrak{R}})\, dv,$$

which may be rewritten as

$$\mathbf{u}(\mathbf{Y}) = ((\mathcal{S}_{\mathfrak{R}}, \mathcal{S}_\mathbf{Y} - \mathcal{S}_\mathbf{Y}'')). \tag{4.4}$$

At this point we can apply the same argument used to obtain (2.29). In fact, if $\mathscr{G}$ is an arbitrary state, by the Cauchy–Schwarz inequality, we have

$$\left|\left(\left(\mathscr{S}_{\Re}-\frac{1}{2}(\mathscr{S}'+\mathscr{S}''),\mathscr{G}\right)\right)\right|\le\left\|\mathscr{S}_{\Re}-\frac{1}{2}(\mathscr{S}'+\mathscr{S}'')\right\|\|\mathscr{G}\|,$$

whence, remembering (3.8), we derive

$$\left|\left(\left(\mathscr{S}_{\Re}-\frac{1}{2}(\mathscr{S}'+\mathscr{S}''),\mathscr{G}\right)\right)\right|\le\frac{1}{2}\|\mathscr{S}'-\mathscr{S}''\|\,\|\mathscr{G}\|. \tag{4.5}$$

This implies that

$$\frac{1}{2}((\mathscr{S}'+\mathscr{S}'',\mathscr{G}))-\frac{1}{2}\|\mathscr{S}'-\mathscr{S}''\|\,\|\mathscr{G}\|\le((\mathscr{S}_{\Re},\mathscr{G}))\le\frac{1}{2}((\mathscr{S}'+\mathscr{S}'',\mathscr{G}))+\frac{1}{2}\|\mathscr{S}'-\mathscr{S}''\|\,\|\mathscr{G}\|,$$

which, putting $\mathscr{C}_0=(1/2)(\mathscr{S}'+\mathscr{S}'')$ and $R_0=(1/2)\|\mathscr{S}'-\mathscr{S}''\|$, becomes

$$((\mathscr{C}_0,\mathscr{G}))-R_0\|\mathscr{G}\|\le((\mathscr{S}_{\Re},\mathscr{G}))\le((\mathscr{C}_0,\mathscr{G}))+R_0\|\mathscr{G}\|. \tag{4.6}$$

Finally, on choosing

$$\mathscr{G}=\mathscr{S}_{\mathrm{Y}}-\mathscr{S}''_{\mathrm{Y}},$$

we obtain the required bound for the displacement at any point of $\boldsymbol{\kappa}(B)$.

Formulae (3.8) and (4.6) are the main result in the extension of the method of two-sided estimates for the solutions of variational inequalities. In despite of the formal analogy with (2.16) and (2.29), their application in particular examples is not immediate because it requires the construction of two admissible states $\mathscr{S}'$ and $\mathscr{S}''$, with $\mathscr{S}''$ depending on $\Re_c$, which is not known *a priori*. The construction of $\Re_c$ involves a knowledge of the solution $\mathscr{S}$ of the linear unconstrained problem.

We wish next to examine in detail the application of our method to the two important problems of the elastic membrane on an obstacle, studied by Lewy and Stampacchia[10], and the Signorini problem of the elastic body on a smooth support, treated, for instance, by Fichera[11].

5. MEMBRANE STRETCHED OVER AN OBSTACLE

Let Ω be a bounded region of the plane and E a subset of Ω contained in the inside of Ω. On E a certain function ψ is defined and we want to determine the equilibrium configuration of an elastic membrane encastred along $\partial\Omega$, the boundary of Ω, and such that the displacement, perpendicular to Ω, is always greater than or equal to ψ (Fig. 4).

We introduce the Hilbert space $\mathscr{M}=H_0^{1}(\Omega)$, defined as the completion with respect the norm

$$\|u\|^2_{H_0^1(\Omega)}=\int_\Omega(u^2+\nabla u\cdot\nabla u)\,dx\,dy,$$

of the space $\mathscr{C}_0^{1}(\Omega)$ of continuously differentiable functions with compact support in Ω. The set

$$\Re\equiv\{v\in H_0^1(\Omega): v(x,y)\ge\psi(x,y)\quad\text{for}\quad(x,y)\in E\} \tag{5.1}$$

is a convex subset of $H_0^1(\Omega)$.

It is well known[9] that the deformation u is the solution to the variational inequality

$$\int_\Omega\nabla u\cdot\nabla(v-u)\,dx\,dy\ge0\qquad\forall v\in\Re. \tag{5.2}$$

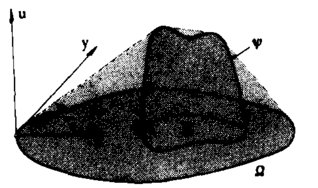

Fig. 4.

Condition under which a solution of (5.2) exists are well known. We seek upper and lower bounds for the deformation strain energy, or more precisely, for the functional

$$((\mathscr{S}_{\mathfrak{R}}, \mathscr{S}_{\mathfrak{R}})) = \int_{\Omega} \nabla u \cdot \nabla u \, dx \, dy, \tag{5.3}$$

where $\mathscr{S}_{\mathfrak{R}}$ denotes the elastic state corresponding to the solution of (5.2). We observe moreover that for $\psi \equiv 0$ (that is, when the obstacle is absent) the solution of the problem is $u \equiv 0$ in Ω. This means that in the unconstrained problem the corresponding elastic state is $\mathscr{S} = \mathbf{O}$.

Any kinematically admissible state $\mathscr{S}'$ is determined from any $u' \in H_0^1(\Omega)$ such that $u'(x, y) \geq \psi(x, y)$ in E.

In order to construct a statically admissible state $\mathscr{S}''$, we consider the set $\mathfrak{R}_c$ of states $\mathscr{S}''$ such that

$$((\mathscr{S}'', \mathscr{S}' - \mathscr{S}'')) \geq 0 \qquad \forall \mathscr{S}' \in \mathfrak{R}. \tag{5.4}$$

This inequality is obtained from (3.5), upon putting $\mathscr{S} = \mathbf{O}$. If u'' is the deformation corresponding to $\mathscr{S}''$ and $\mathbf{p}'' = \nabla u''$ is its gradient, we may write (5.4) in the form

$$\int_{\Omega} \mathbf{p}'' \cdot \nabla u' \, dx \, dy \geq \int_{\Omega} \mathbf{p}'' \cdot \mathbf{p}'' \, dx \, dy \qquad \forall u' \in \mathfrak{R}.$$

On integrating by parts in the left-hand term and remembering that $u' \in H_0^1(\Omega)$, we obtain

$$-\int_{\Omega} \operatorname{div} \mathbf{p}'' u' \, dx \, dy \geq \int_{\Omega} \mathbf{p}'' \cdot \mathbf{p}'' \, dx \, dy \qquad \forall u' \in \mathfrak{R}. \tag{5.5}$$

We now choose $\mathbf{p}''$ such that $\operatorname{div} \mathbf{p}'' \leq 0$ in Ω. Then (5.5) is satisfied *a fortiori* provided $\mathbf{p}''$ satisfies the inequality

$$-\int_{\Omega} \operatorname{div} \mathbf{p}'' \max(0, \psi) \, dx \, dy \geq \int_{\Omega} \mathbf{p}'' \cdot \mathbf{p}'' \, dx \, dy. \tag{5.6}$$

For instance, on taking $\mathbf{p}'' \equiv (-\alpha x, -\alpha y)$, with α a positive constant, we see that (5.6) easily becomes

$$2\alpha \int_{\Omega} \max(0, \psi) \, dx \, dy \geq \alpha^2 \int_{\Omega} (x^2 + y^2) \, dx \, dy = \alpha^2 J_p$$

where J_p is the polar moment of Ω with respect the origin of the coordinates. Therefore any α such that

$$\alpha \leq \frac{2}{J_p} \int_{\Omega} \max(0, \psi) \, dx \, dy \tag{5.7}$$

allows us to completely define $\mathbf{p}''$ and hence $\mathscr{S}''$. We can now apply (3.8) to obtain bound for $\|\mathscr{S}_{\mathfrak{R}}\|^2$. To get pointwise bounds, we denote by x_1, y_1 the coordinates of the given point. As it is known, the function

$$u_Y = \frac{1}{2\pi}\ln r = \frac{1}{2\pi}\ln\sqrt{((x-x_1)^2+(y-y_1)^2)}$$

represents a fundamental solution for the Laplace equation. Let $\mathcal{S}_Y$ be the associated elastic state and let us introduce an auxiliary state $\mathcal{S}''_Y$, defined by the vector $\mathbf{p}''_Y$ where

$$\operatorname{div}\mathbf{p}''_Y = 0 \quad \text{in} \quad \Omega,$$

$$\mathbf{p}''_Y\cdot\mathbf{n} = \nabla u''_Y\cdot\mathbf{n} = \frac{1}{2\pi}\frac{\partial}{\partial n}(\ln r) \quad \text{on} \quad \partial\Omega.$$

Thus, applying (4.4) directly, we can write

$$\begin{aligned} u(x_1,y_1) &= ((\mathcal{S}_{\mathfrak{M}}, \mathcal{S}_Y - \mathcal{S}''_Y)) \\ &= \int_\Omega \nabla u_{\mathfrak{M}}\cdot\left(\frac{1}{2\pi}\nabla\ln r - \mathbf{p}''_Y\right)dx\,dy, \end{aligned} \tag{5.8}$$

whence we arrive easily at (4.6).

Example 5.1. Let Ω be a circle of radius R and E a concentric circle of radius $(1/2)R$. In E, a function ψ is defined which is supposed to be continuous and strictly positive (Fig. 5). We denote by M and m the maximum and minimum values (both positive) of ψ. In this case, referred to polar coordinates whit origin at the centre of Ω, the functions u' assumes the form:

$$u'(r) = \begin{cases} M & \text{for} \quad 0\le r\le(1/2)R, \\ 2M(1-(r/R)) & \text{for} \quad (1/2)R\le r\le R, \end{cases}$$

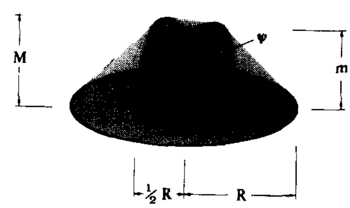

Fig. 5.

which clearly defines a kinematically admissible state. On the other hand, on choosing $\mathbf{p}''\equiv(-\alpha x, -\alpha y)$, where

$$\alpha = \frac{2}{J_p}\,m\ \text{meas}\ E = \frac{m}{R^2},$$

we automatically satisfy (5.7) hand so have a complementary admissible state.

Finally, if we want to bound the deformation in the centre of Ω, we must construct a vector $\mathbf{p}''_O$ such that

$$\operatorname{div}\mathbf{p}''_O = 0 \quad \text{in} \quad \Omega,$$

$$\mathbf{p}''_O\cdot\mathbf{n} = \frac{1}{2\pi}\left[\frac{\partial}{\partial r}(\ln r)\right]_{r=R} = \frac{1}{2\pi R}.$$

We easily see [6] that

$$\mathbf{p}''_O \equiv \left(\frac{1}{2\pi}xr^{-2}, \frac{1}{2\pi}yr^{-2}\right)$$

satisfies the required conditions, and hence may be used in (5.8) and (4.6).

6. THE PROBLEM OF THE ELASTIC BODY SUPPORTED BY A SMOOTH PLANE

Let $\boldsymbol{\kappa}(B)$ be the reference configuration of an elastic body loaded in the following way (Fig. 6): the boundary $\partial\boldsymbol{\kappa}(B)$ consists of a part $\partial_2\boldsymbol{\kappa}(B)$ on which surface tractions are assigned, together with its complement $\overline{\partial_1\boldsymbol{\kappa}(B)}$ which is in contact with a smooth rigid support, preventing any displacement in the normal direction.†

We assume a system of cartesian coordinates with origin taken at a point in the supporting plane, and with the X_3-axis along the outer normal, and the X_1X_2-axes placed in this plane.

The solutions of the problem are then constrained by the condition

$$u_3 = \mathbf{u}\cdot\mathbf{n} \geq 0 \quad \text{on} \quad \overline{\partial_1\boldsymbol{\kappa}(B)}. \tag{6.1}$$

We seek the solutions in the Hilbert space $H^1(\boldsymbol{\kappa}(B))$ defined as the completion of functions $C^1(\overline{\boldsymbol{\kappa}(B)})$ with respect the norm

$$\|\mathbf{u}\|^2 = \int_{\boldsymbol{\kappa}(B)} [\mathbf{u}\cdot\mathbf{u} + tr(\nabla\mathbf{u}\nabla\mathbf{u}^T)]\,\mathrm{d}v. \tag{6.2}$$

On discarding all rigid motions compatible with the constraints, we see that the norm (6.2) is equivalent to the norm defined on the deformation energy, given by

$$\|\mathcal{S}\|^2 = \int_{\boldsymbol{\kappa}(B)} tr(\mathbf{T}\tilde{\mathbf{E}})\,\mathrm{d}v. \tag{6.3}$$

The set

$$\mathfrak{R} \equiv \{\mathbf{u}\in H_0^1(\boldsymbol{\kappa}(B)) : \mathbf{u}\cdot\mathbf{n} = u_3 \geq 0 \quad \text{on} \quad \overline{\partial_1\boldsymbol{\kappa}(B)}\} \tag{6.4}$$

is a closed convex subset of $H^1(\boldsymbol{\kappa}(B))$.

The Signorini problem is therefore reduced to the determination of a vector $\mathbf{u}_{\mathfrak{R}} \in \mathfrak{R}$ such that

$$\int_{\boldsymbol{\kappa}(B)} tr\{\mathbf{T}_{\mathfrak{R}}(\tilde{\mathbf{E}}' - \tilde{\mathbf{E}}_{\mathfrak{R}})\}\,\mathrm{d}v \geq \int_{\partial_2\boldsymbol{\kappa}(B)} \hat{\mathbf{t}}\cdot(\mathbf{u}' - \mathbf{u}_{\mathfrak{R}})\,\mathrm{d}s \quad \forall\mathbf{u}'\in\mathfrak{R}. \tag{6.5}$$

To find bounds on the energy and on the solution at a point we must construct the admissible states entering into (3.8) and (4.6).

A kinematically admissible state $\mathbf{u}'$ is, for instance, defined by the displacement field

$$\mathbf{u}' = \begin{cases} u_1 = u_2 = 0 & \text{in} \quad \boldsymbol{\kappa}(B), \\ u_3 = \alpha X_3 & \text{in} \quad \boldsymbol{\kappa}(B), \end{cases} \tag{6.6}$$

where α is a positive constant, small enough compared to unity for the hypotheses of the classical theory to apply.

On the other hand, for the state $\mathcal{S}''$, we observe that it must belong to the convex set $\mathfrak{R}_c$ defined by (3.5). In our case this inequality becomes

$$\int_{\boldsymbol{\kappa}(B)} tr\{\mathbf{T}''(\tilde{\mathbf{E}}' - \tilde{\mathbf{E}}'')\}\,\mathrm{d}v \geq \int_{\boldsymbol{\kappa}(B)} tr\{\mathbf{T}(\tilde{\mathbf{E}}' - \tilde{\mathbf{E}}'')\}\,\mathrm{d}v \quad \forall\mathbf{u}'\in\mathfrak{R}, \tag{6.7}$$

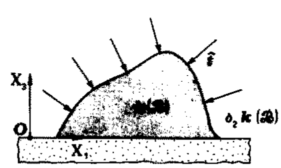

Fig. 6.

†Of course many variants and generalizations of this problem are possible (*cf.* for instance Duvaut and Lions[12]).

where $\mathbf{T}$ is the stress field in the (linear) problem associated which tractions $\hat{\mathbf{t}}$ on $\partial_2\boldsymbol{\kappa}(B)$ and displacements $u_3 = 0$ on $\overline{\partial_1\boldsymbol{\kappa}(B)}$. The strain energy corresponding to $\mathbf{T}$ may be bounded using classical methods.

If we require $\mathbf{T}''$ to satisfy the static conditions:

$$\operatorname{div}\mathbf{T}'' = \mathbf{O} \quad \text{in} \quad \boldsymbol{\kappa}(B),$$
$$\mathbf{T}''\mathbf{n} = \hat{\mathbf{t}} \quad \text{on} \quad \partial_2\boldsymbol{\kappa}(B),$$
$$\mathbf{T}''\mathbf{s} = \mathbf{O} \quad \text{on} \quad \overline{\partial_1\boldsymbol{\kappa}(B)},$$

where $\mathbf{s}$ is a generic unit vector in the plane $X_3 = 0$, then, by applying the Gauss–Green formula to (6.7) and remembering that $\tilde{\mathbf{E}}' = (1/2)(\nabla\mathbf{u}' + \nabla\mathbf{u}'^T)$, we get

$$\int_{\boldsymbol{\kappa}(B)} tr(\mathbf{T}''\tilde{\mathbf{E}}'')\,\mathrm{d}v \le \int_{\boldsymbol{\kappa}(B)} tr(\mathbf{T}\tilde{\mathbf{E}}'')\,\mathrm{d}v. \tag{6.8}$$

Thus, (6.8) is a further restriction on $\mathbf{T}''$ in order that $\mathcal{S}''$ belongs to $\mathfrak{R}_c$. We can also write (6.8) in the form

$$\|\mathcal{S}''\|^2 = ((\mathcal{S}'', \mathcal{S}'')) \le ((\mathcal{S}, \mathcal{S}'')), \tag{6.9}$$

which shows that *it is not necessary to determine* $\mathcal{S}$ *to ensure that* $\mathcal{S}''$ *satisfies* (6.9). It suffices to use (2.29) with $\|\mathcal{S}''\|$ less than the lower bound of $((\mathcal{S}, \mathcal{S}''))$ given by (2.29).

Pointwise bounds for the solution may be calculated with the help of formula (4.4). For this, we must construct an auxiliary state $\mathcal{S}''_Y$ satisfying conditions (1) and (2) given in Section 4. The construction of this auxiliary function requires the same operations as in the linear problems.

7. CONCLUSIONS

The method of two-sided bounds for the energy of solutions of variational inequalities consists essentially of three steps:

(1) To define two convex subsets of the function space, in which the problem is formulated in such a way that the solution is the only common element.

(2) To construct two admissible states, each belonging to one of the two subspaces.

(3) To find an auxiliary state, which, with the help of a fundamental solution and Betti's theorem, produces pointwise bounds.

These operations are formally similar to those used in bounding the solutions to the linear boundary value problems. Now however, the two orthogonal subspaces are replaced by two complementary convex subsets.

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